The solitary wave on a stream with an arbitrary distribution of vorticity

By T. BROOKE BENJAMIN
Department of Engineering, University of Cambridge

(Received 17 April 1961)

The theoretical work reported herein makes a departure from the many previous analyses of the solitary wave which have treated the wave as an example of irrotational fluid motion. The present analysis is of more general scope in that it covers the whole category of examples in which the wave may propagate in either direction on a horizontal stream whose primary velocity distribution $U(y)$ is an arbitrary function (i.e. there is no restriction on the extent of the variations of $U(y)$). An approximate form of the wave profile is found in general to be $a\,\mathrm{sech}^2\{(x-ct)/b\}$, as it is according to previous theories applicable to the wave on a uniform stream, but the relationships amongst the wave amplitude a, the length scale b, and the two propagation velocities c (positive downstream and negative upstream) depend in complicated fashion on the form of $U(y)$.

1. Introduction

As far as is known, all of the many previous theoretical investigations of the solitary wave in a homogeneous fluid have proceeded on the assumption of irrotational motion.† Analyses made on this basis may apply very accurately to a real solitary wave advancing into still water, because then the effect of viscosity on the fluid motion generated by passage of the wave may be confined to thin boundary layers on the channel bottom and sides. For a wave that occurs on running water, however, the assumption of irrotational motion is essentially artificial since the velocity of the stream is thereby implied to be the same at all depths: that is, the effects of friction on the primary flow have to be ignored, as well as its effects on the wave.

As a contribution to the understanding of the latter case, the analysis presented in this paper covers the problem of a solitary wave propagating along a non-uniform parallel stream, such as might develop in a long channel when vorticity produced by frictional action at the boundary becomes diffused over the whole cross-section. The analysis in fact deals comprehensively with the generalized theoretical problem in which the distribution of horizontal velocity with depth is arbitrary, being unrestricted both as to form and as to the magnitude of the velocity variations with depth. A two-dimensional problem is formulated,

† Peters & Stoker (1960) have investigated the solitary wave in a liquid with continuous density stratification, so dealing with a case where the wave motion is rotational. In their theoretical model the only vorticity is that generated by the wave, and so the basic difficulty of the present problem is not encountered. Nevertheless some resemblances between their analysis and the present one may be noted.

relating practically to a rectangular channel whose breadth considerably exceeds the depth of water; and justification for a frictionless theoretical model is given in § 2. The equations of motion and boundary conditions respective to a rotational flow with a free surface under gravity are solved to an approximation which comprises, in effect, a 'second-order shallow-water theory' (cf. Stoker 1957, p. 343), and which therefore compares with the approximations to the 'irrotational' solitary wave found originally by Boussinesq (1871) and Rayleigh (1876), and alternatively derived by Keulegan & Patterson (1940), Keller (1948), Stoker (1957, § 10.9) and others. The method of solution is open to the development of successive further approximations, even though the work would become much more complicated than the fairly simple task carried through here; and the precise view that the methods affords regarding the magnitude of the residual error gives security to the present approximation. Since in this last respect the theory appears to present essentially the same situation as does the theory of the irrotational solitary wave, the question of convergence does not seem particularly urgent. The higher-order approximations that have been found on the basis of potential theory, for instance by McCowan (1891), Weinstein (1926), Packham (1952) and Long (1956), all strongly suggest the convergence of approximate methods when the wave amplitude is sufficiently small, and a rigorous proof of convergence for sufficiently small amplitudes—hence a proof that a solitary wave of strictly permanent form really exists mathematically—was established by Friedrichs & Hyers (1954).

When approached by orthodox perturbation methods, problems concerning surface waves on rotational flows are generally rather difficult; and the very small numbers of papers on the subject contrasts remarkably with the vast literature on irrotational wave motions. Notably, Thompson (1949) and Biésel (1950) have investigated the relation between the propagation speed and wavelength of infinitesimal sinusoidal waves, and Burns (1953) has demonstrated various properties of waves whose lengths are extremely long compared with the depth of the stream along which they are supposed to travel. In particular, Burns obtained a general formula for the speed of infinitesimal long waves, and, as might be expected, this formula is recovered by the present analysis upon imposing a condition that the amplitude of the solitary wave becomes indefinitely small, which also makes its length indefinitely large [as a property in common with irrotational solitary waves (cf. Ursell 1953 or Benjamin & Lighthill 1954, p. 449), the parameter $amplitude \times (length)^2 \div (depth)^3$ is found to be approximately a constant, in fact not much different from unity, over the whole spectrum of solitary waves possible upon a stream with given non-uniform velocity distribution]. However, the methods previously applied to waves on rotational flows appear to offer little possibility of extension to the present problem, least of all when it is generalized to arbitrary vorticity distributions, and a novel method of analysis has had to be found. As regards its essentials which are explained in the first part of § 4, the method may have some general interest outside its present context, since evidently it might be useful in other practical problems concerned with disturbed rotational flows (one very simple incidental application is mentioned in a footnote).

There is, nevertheless, one example of the present general problem that admits treatment by an adaptation of existing methods: this is the case of a solitary wave on a stream with *constant* vorticity. An analysis based on an extension of Rayleigh's method (1876) is outlined in an appendix, and its results are compared with those according to the general analysis given in the body of the paper.

2. Preliminary considerations

In the theory friction and turbulence are not considered, except implicitly in so far as they determine the primary velocity distribution, and the stream is represented as a laminar flow of incompressible inviscid fluid having a prescribed distribution of total head and vorticity over the streamlines. Although in a real fluid the velocity must vanish on the channel bottom, and the rate of shearing must vanish at the free surface, these conditions do not have to be applied to the frictionless model; they can be optionally allowed in specifying the primary flow —provided the difficulty explained below regarding the first condition does not arise—but they need to be relaxed in the analysis of the disturbed flow under a wave. As regards the applicability of the theory in practice, the essential assumption is that the length scale of the wave motion is considerably smaller than the lengths to be associated with 'changes in boundary-layer structure' (i.e. the lengths over which changes in flow conditions may be accommodated by adjustments of the primary velocity distribution by frictional action), so that the dynamical effects of the wave are principally those due to distortion of the prevailing vorticity field. This principle on which short-scale localized motions in a shearing flow—whose vorticity owes in the first place to friction—may be treated as frictionless is a familiar and well-tried one, and it can be applied with a fair degree of confidence to the present problem.

Accepting the existence of the solitary wave as a solution to the corresponding problem of irrotational motion, one may readily comprehend the possibility of a similar wave on a non-uniform stream, provided the flow is more than marginally *subcritical* (i.e. mean velocity $< C_0$, where $C_0 = \sqrt{(g \times \text{depth})}$ is the speed, relative to the mean flow, of infinitesimal waves of extreme length). The special attribute of the subcritical case is that a solitary wave, having a relative speed at least as large as C_0, will travel either upstream or downstream at a fair speed relative to every one of the strata within the primary flow. That is to say, in a frame of reference travelling with the wave the apparent velocity of the fluid will be of the same sign, and not particularly small, at all depths (e.g. as indicated in figure 1). Hence, considering the state of steady motion observed in the moving frame of reference, one can see how the dynamical conditions for a steady wave might be satisfied in much the same way as for the corresponding irrotational flow: instead of the condition of zero vorticity everywhere, there is now the condition that the vorticity along each streamline should remain constant, and the condition of constant total head along the *free* streamline is the same as before even though the respective value differs from the values of total head on the other streamlines.

A difficulty arises, however, if the stream is near the critical condition and if the primary velocity distribution $U(y)$ falls to zero or to a small value at the channel bottom $y = 0$. The absolute velocity c_- of a wave propagating against the stream may then be quite small in comparison with the mean flow velocity, or may be made negative (i.e. the wave may be carried downstream) if the flow is supercritical. In this case the velocity $W(y) = c_- + U(y)$ of the fluid relative to the wave may be very small, or may become negative, near the channel bottom; consequently it may be impossible to satisfy the condition of constant total head on streamlines near the bottom. This matter may be understood more clearly

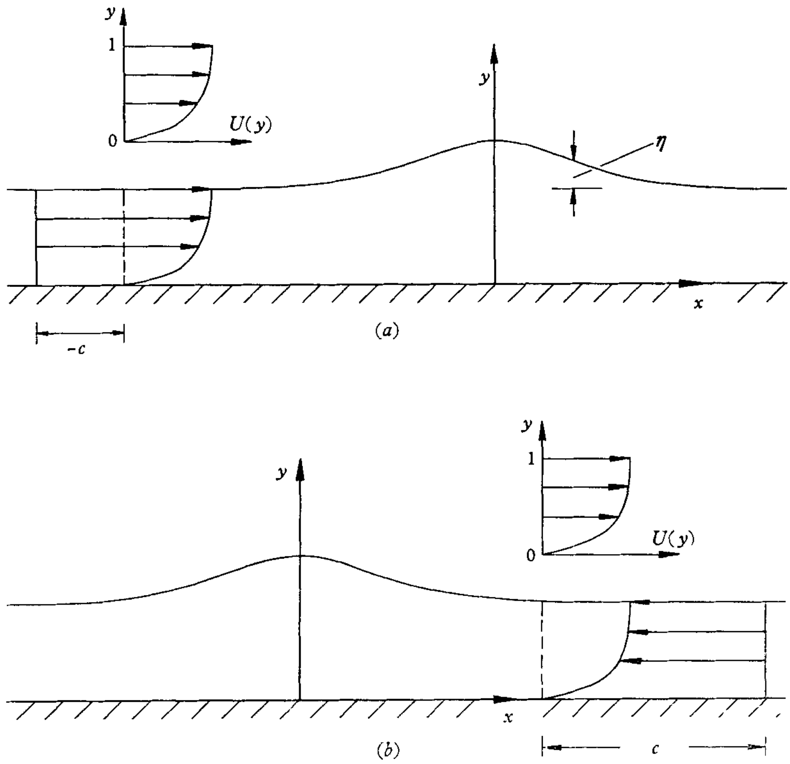

FIGURE 1. Definition sketches showing the velocity profile of the primary flow measured with respect to axes moving with the wave: (a) upstream propagation with $c < 0$; (b) downstream propagation with $c > 0$.

by supposing, as a first approximation, that the pressure p in the fluid is everywhere the hydrostatic value. Correspondingly, at all depths below a point on the free surface which is raised a height η by the wave, the 'piezometric head' $(p/\rho) + gy$ is increased by an amount $g\eta$ above its value in the undisturbed stream. To maintain constant total head with respect to the moving frame of reference, this increase must be balanced by a reduction in velocity head, which is of course a positive definite quantity (i.e. with lower limit 0^+ at stagnation points). We see therefore that a steady wave becomes impossible—at least without a region of stagnation within the flow—if at any depth the initial velocity head $\frac{1}{2}W^2(y)$ is smaller than $g\eta_m$, where η_m is the maximum elevation of the wave; and as an extreme case, no wave is possible for which $W(y)$ vanishes at any depth—a conclusion that was reached by Burns (1953) in his study of infinitesimal waves.

For example, no standing wave ($c_- = 0$) would be found for a stream with $U(0) = 0$, although according to the irrotational-flow theory standing waves definitely exist over a certain range of supercritical uniform flows.

In such cases where a continuous steady flow appears to be ruled out, it might still be possible to find a steady wave relative to which the flow separates from the channel bottom and there is a region of stagnant fluid (i.e. the region is carried along by a travelling wave). In practice separation probably does sometimes occur under fairly large standing waves, or under slowly moving ones, and so a study of the stagnation effects in question might have some practical bearing. However, this rather special aspect of the over-all problem does not warrant further attention in this paper, where the aim is to contend with the more general difficulties of analysing the solitary wave on a non-uniform stream. The theory is therefore developed under the restriction that at all depths $W(y)$ is sufficiently large to avoid stagnation effects.

3. Formulation of the mathematical problem

We adopt a scheme of dimensionless variables in which the depth h of the undisturbed stream is taken as the unit of length and a certain datum U_0, which need not be specified explicitly, is taken as the unit of velocity. Thus, for instance, the co-ordinates (x, y) defined in figure 1 measure distances as multiples of h, and all variable velocities are symbolized as multiples of U_0. The symbol g here denotes the gravitational acceleration as a multiple of U_0^2/h; that is, $g = g^*h/U_0^2$ if g^* is the dimensional quantity.

The wave is supposed to occur upon a horizontal stream in which, when undisturbed by the wave, the velocity distribution is $U(y)$ everywhere (see figure 1). On the assumption that the wave propagates without change of form, the problem can be treated as one of steady motion by taking axes (x, y) moving horizontally with the wave. Thus, letting c denote the absolute velocity of propagation downstream, we have that the primary fluid velocity in the moving frame of reference is $W(y) = U(y) - c$ in the direction of x. For a wave propagating *upstream*, as indicated in figure 1(*a*), c is negative according to this definition, so that $W(y)$ is the sum of two positive components. In the case of *downstream* propagation, $W(y)$ will be negative as indicated in figure 1(*b*). In the development of the theory there is no need to distinguish between these two cases, each of which is covered by the appropriate evaluation of $W(y)$ in the final results.

We introduce a stream-function ψ in terms of which the velocity components u and v, parallel to x and y respectively, and the vorticity ζ are given by

$$\left.\begin{aligned} u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \\ \text{and} \qquad \zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \end{aligned}\right\} \tag{1}$$

The bottom of the channel may be defined as the streamline $\psi = 0$, and the kinematical condition for the wave profile to be stationary is that the free surface is also a streamline. If $\eta(x)$ denotes the elevation of the wave above the level of the undisturbed free surface, i.e. above $y = 1$, the kinematical surface condition may be expressed by the equation

$$\psi(x, 1+\eta) = \Psi(1), \tag{2}$$

of which the right-hand side is the value at $y = 1$ of the known stream-function

$$\Psi(y) = \int_0^y W(y)\,dy \tag{3}$$

for the primary flow. To specify that the wave must be a solitary wave, not for instance a train of periodic waves, we impose the further condition that

$$\eta(x) \to 0, \quad \psi(x,y) \to \Psi(y) \quad \text{for} \quad x \to \pm\infty. \tag{4}$$

The dynamical condition for steady motion is that the vorticity must be constant along each streamline (Lamb 1932, p. 244), and here this condition may be expressed formally as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta_0(\psi), \tag{5}$$

where ζ_0 is the function defined by the property that $\zeta_0(\Psi)$ gives the vorticity of the primary stream. [Note that although this function is determined exactly by any assumed velocity distribution $W(y)$, it cannot be worked out explicitly except in a few particularly simple cases. For example, if

$$W(y) = c + (1+k)\,\overline{U}y^k,$$

which is a case discussed later, we have $\Psi = cy + \overline{U}y^{k+1}$ and $\zeta_0 = k(k+1)\,\overline{U}y^{k-1}$; thus, for $k = \frac{1}{7}$ and $c \neq 0$ as considered later, the task in question is clearly impossible.]

The total head H defined as follows is also constant along each streamline (Lamb 1932, p. 21):

$$H(\psi) = \tfrac{1}{2}(u^2+v^2) + p + gy. \tag{6}$$

(Here p is the pressure measured in units of U_0^2 times the density of the fluid.) By reference to the Eulerian equations of motion, from which both (5) and (6) are obtained by integration, it is easily seen that the property expressed by (6) is either entirely concomitant with (5) or at most, as in our case, requires an additional condition to be satisfied on only one streamline. If the upper boundary of the fluid were solid instead of free, then (6) would be satisfied automatically in consequence of (5) being satisfied, essentially because the flow is then determined completely by the kinematical conditions at known boundaries and the pressure can adjust to the values determined by (6). In the present problem, however, the position of the wave surface is not known initially; and since it is a free surface, the wave is dynamically possible only if the solution to (5) makes the pressure there constant. Hence, expressing the fact that both H and p are the same everywhere on the surface, including points so remote from the wave that $y \to 1$ and $(u^2+v^2) \to W^2(1)$, we get from (6) the dynamical surface condition

$$W^2(1) - (u^2+v^2)_{y=1+\eta} = 2g\eta(x). \tag{7}$$

Equation (5) for ψ together with the boundary conditions (2), (4) and (7) [also $\psi(0) = 0$] define the mathematical problem. In previous theories of the solitary wave, it has been assumed that $W = \text{const.}$ and therefore $\zeta_0 = 0$. Equation (5) confirms that ψ is then a harmonic function, being in fact the conjugate of a velocity potential, and this property has been used in various ways to derive approximations to the wave of finite amplitude. None of the previous methods appears capable of straightforward extension to account for an arbitrary distribution of vorticity, and the following new approach has had to be devised.

4. Analysis

By the assumption explained at the end of § 2, $W(y)$ has the same sign throughout the range $0 \leqslant y \leqslant 1$, and therefore $\Psi(y)$ is a monotonically varying function; furthermore, the bounding streamline $\psi = 0$ is assumed to remain attached to the bottom. Hence, over any vertical section x through the flow, the height of the streamlines can be taken to be a single-valued function of ψ. Accordingly, the whole flow can be represented uniquely by the transformation

$$y = y(x, \psi). \tag{8}$$

Introducing a suffix notation for partial differentiation, we now have that the velocity components are $u = 1/y_\psi$, $v = y_x/y_\psi$, and the corresponding expression for the vorticity is readily found to be

$$\zeta = -\frac{1}{y_\psi^3}[y_{xx}y_\psi^2 - 2y_{x\psi}y_x y_\psi + y_{\psi\psi}(1+y_x^2)]. \tag{9}$$

Setting this expression equal to $\zeta_0(\psi)$, as in equation (5), we obtain an equation for $y(x, \psi)$ which, together with the boundary conditions appropriately transformed, comprises an alternative statement of the mathematical problem. Like (5), this is a second-order non-linear equation of first degree; but the equation is inherently simpler than (5), despite its more complex form, since the coefficient $\zeta_0(\psi)$ is now a function of an independent variable.

The equation for $y(x, \psi)$ is still hardly tractable, however, and the progress of the analysis depends essentially on the next step, which is to transform the independent variable ψ. We write

$$\psi = \Psi(s), \tag{10}$$

where Ψ is the stream-function for the primary flow, as defined by (3). Thus, the new variable s, which varies between zero on the bottom and unity on the free surface, is equal to the height which the respective streamline approaches asymptotically at the outskirts of the wave; and since Ψ varies monotonically and there is no separation, the value of s specifies ψ uniquely. Using the fact that $d\psi/ds = W(s)$ by definition, we get at once for the velocity components and the vorticity

$$\left.\begin{aligned} &u = W(s)\frac{1}{y_s}, \quad v = W(s)\frac{y_x}{y_s}, \\ &\zeta = -\frac{1}{y_s^3}[W(s)\{y_{xx}y_s^2 - 2y_{xs}y_x y_s + y_{ss}(1+y_x^2)\} - W'(s)\,y_s(1+y_x^2)]. \end{aligned}\right\} \tag{11}$$

Now, since the streamlines are determined by s alone, and since $s = y$ in the primary flow infinitely remote from the wave, a simple equivalent of the dynamical equation (5) is therefore

$$\zeta = W'(s); \tag{12}$$

that is, we have $W'(s) = \zeta_0(\psi)$ to express the property that the initial vorticity $W'(y)$ is conserved along each streamline as it is displaced by the wave from its initial height y. Equating $W'(s)$ to the previous expression for ζ and multiplying by y_s^3 to eliminate fractions, we get as the equation for $y(x,s)$†

$$W(s)\{y_{xx}y_s^2 - 2y_{xs}y_xy_s + y_{ss}(1+y_x^2)\} + W'(s)\{y_s^3 - y_s(1+y_x^2)\} = 0. \tag{13}$$

As is to be expected, a particular integral of this equation is $y = s$, corresponding to the primary flow.

The problem is thus reduced to finding the solution $y(x,s)$ of (13) subject to the following boundary conditions: first, the kinematical conditions

$$y(x,0) = 0, \quad y(x,1) = 1 + \eta(x); \tag{14}$$

secondly, the asymptotic conditions corresponding to (4)

$$\eta(x) \to 0, \quad y(x,s) \to s \quad \text{for} \quad x \to \pm\infty; \tag{15}$$

and, thirdly, the dynamical surface condition (7) which, by substitution from (11), now becomes

$$W^2(1)\left\{1 - \left(\frac{1+y_x^2}{y_s^2}\right)_{s=1}\right\} = 2g\eta(x). \tag{16}$$

If it were required to find explicitly the whole velocity field under the wave, the solution $y(x,s)$ would need to be inverted to give $s(x,y)$, which upon substitution into (10) would give $\psi(x,y)$. However, this step is unnecessary at present where the object is only to find the wave profile.

We first observe that if the x-variations of y were sufficiently small, equation (13) would reduce to

$$W(s)y_{ss} + W'(s)\{y_s^3 - y_s\} = 0, \tag{17}$$

a second-order ordinary differential equation whose general solution is easily found. The solution satisfying the condition $y = 0$ for $s = 0$ is

$$y = \int_0^s \frac{W(s)\,ds}{[W^2(s) - 2A]^{\frac{1}{2}}}, \tag{18}$$

where $2A$ is an arbitrary constant—or a very slowly varying function of x if (18) is to be interpreted as an approximate solution of (13).‡ This expression

† It is worth noting incidentally that the equivalent equation for $s(x,y)$ is

$$W(s)\{s_{xx} + s_{yy}\} + W'(s)\{s_x^2 + s_y^2 - 1\} = 0.$$

[This is a special case of an equation derived by Long (1953, equation (12) for y_0); his equation allows for density stratification in the fluid and reduces to the present form when density is made constant.] Though neater than (13), this equation presents the grave difficulty that the coefficients $W(s)$, $W'(s)$, which are to remain arbitrary in the analysis of the general problem, are functions of the dependent variable. Evidently (13) rather than this equation will provide the more tractable basis for developing a general method of solution by successive approximation. Note that in the case of uniform flow with $W = C$, say, we have simply $\Psi = Cy$ which means that $s = \psi/C$, and so the present equation merely reproduces Laplace's equation for ψ.

‡ Note that the second arbitrary constant arising in the general solution of (17) is simply an additional term to (18). An interesting precise application of this solution is to the problem where a stream is at first confined between two rigid parallel boundaries at $y = 0$ and $y = 1$, and thereafter flows through a contraction or expansion into the space between two other such boundaries, say at $y = a$ and $y = b$, not necessarily the same distance apart as the first. For the parallel flow well downstream from the connexion, the choice of the two arbitrary constants in general allows both the boundary conditions $y = a$ for $s = 0$ and $y = b$ for $s = 1$ to be satisfied, and the solution then describes the spatial redistribution of the streamlines according to the dynamical requirement that the original vorticity distribution amongst them should be preserved. If defined as in (18), the constant A is then the gain in piezometric head between the initial and final parallel flows.

cannot be made to satisfy both the kinematical condition $y = 1+\eta$ for $s = 1$ and the dynamical condition (16); which accords with the well-known fact that no wave of extreme length can have both finite amplitude and permanent form. We note, however, that (16) alone would specify $A = g\eta$, and this suggests the following scheme for finding an approximate solution to the complete problem. The possibility of a solution in the form

$$y = \int_0^s \frac{W(s)\,ds}{[W^2(s)-2A(x)]^{\frac{1}{2}}} + B(x,s) + \ldots \tag{19}$$

is considered. Then, assuming the wave to be fairly long compared with depth and everywhere smooth, so that its slope $\eta'(x)$ is nowhere larger than a small fraction ϵ of $\eta(x)$ and that the successive differential coefficients $\eta', \eta'', \ldots$ diminish in order of magnitude at least as rapidly as integral powers of ϵ, we see from (16) that $A = g\eta + O(B)$ and hence, from (13), that B will not be greater than $O(\eta'')$.

[At this point the matter considered towards the end of §2 deserves to be recalled. For the expression (19) to be meaningful, it is necessary to assume that $W^2(s) > 2A$ throughout the range $0 < s < 1$, and evidently this condition corresponds to the avoidance of a stagnation effect as explained in §2. According to the approximation (18), the velocity head along any streamline defined by s is precisely $\frac{1}{2}W^2(s) - A$; and since $A = g\eta$ by (16), which means simply that the pressure is hydrostatic to this approximation, the conclusions of §2 are reproduced exactly. The approximation represented by (19) does not admit such a precise physical interpretation; but it would seem that the assumption $W^2(s) > 2A$ is still essentially an expression of the physical assumption made in the last paragraph of §2.]

To put the scheme of approximation on a more definite basis, a well-established property of the irrotational solitary wave may be taken as a guide. It is reasonable to proceed on the assumption that this property also holds generally in the present case, and this can be checked *a posteriori* when an approximation to the wave profile is finally established. Let a denote the wave amplitude, i.e. the maximum of the elevation η; and let b denote a measure of the length scale of the wave such that $b \equiv 1/\epsilon$, where ϵ is the small quantity specified above. Then, as the special property of solitary waves, we have that $ab^2 = O(1)$. [This important property was first recognized by Korteweg & de Vries (1895). More recently Ursell (1953) and Benjamin & Lighthill (1954) have discussed its various implications, in particular its bearing on the matter of reconciling the existence of solitary waves with the prediction of 'non-linear shallow-water theory' (e.g. see Stoker 1957, chap. 10) that positive long waves tend to steepen ahead of their crests and eventually form bores.] As in basic theories of the irrotational solitary wave (e.g. those of Boussinesq 1871; Rayleigh 1876; Keulegan & Patterson 1940; or Stoker 1957, §10.9), the smallest terms to be retained in the present approximation are $O(a\epsilon^2)$; and eventually, in accord with the property just mentioned, the same standing is given to $O(a)$ and $O(\epsilon^2)$. In other words, we follow the overall scheme indicated by

$$\eta'' = O(a\epsilon^2) = O(a^2), \quad \eta'^2 = O(a^2\epsilon^2) = O(a^3), \quad \text{etc.},$$

and develop, in effect, a second-order approximation in a.

Substituting (19), we find that (13) is satisfied to the required order of approximation if

$$W(s)\{B_{ss}+A''(x)I_2(s)\}+2W'(s)B_s = 0, \tag{20}$$

where

$$I_2(s) = \int_0^s \frac{dZ}{W^2(Z)}. \tag{21}$$

First and second integrals of (20) can be obtained immediately: thus we get

$$B(x,s) = -A''(x)\int^s W^{-2}(X)\left\{\int^X W^2(Y)I_2(Y)\,dY\right\}dX. \tag{22}$$

Hence the approximate solution of (13) satisfying the conditions $y(x,0) = 0$ and $\zeta(x, 0) = W'(0)$ is seen to be

$$y(x,s) = \int_0^s \frac{W(s)\,ds}{[W^2(s)-2A(x)]^{\frac{1}{2}}} - \alpha(s)A''(x), \tag{23}$$

where

$$\alpha(s) = \int_0^s\int_0^X\int_0^Y \frac{W^2(Y)}{W^2(X)\,W^2(Z)}\,dX\,dY\,dZ, \tag{24}$$

and where it is supposed that A has, in common with η, the property $A(x)\to 0$ for $x\to \pm\infty$.

The rest of the analysis is very straightforward in principle. The remaining boundary conditions to be satisfied are the two at the free surface, which constitute two relationships between the unknown functions $A(x)$ and $\eta(x)$. Elimination of $A(x)$ then gives an equation for the surface elevation $\eta(x)$.

First, putting (23) into the dynamical boundary condition (16), we obtain, to the order of approximation defined above,

$$A - W^2(1)\,\alpha'(1)\,A'' = g\eta. \tag{25}$$

To the same order, this is equivalent to

$$A = g\{\eta+\beta(1)\,\eta''\}, \tag{26}$$

with

$$\beta(1) = W^2(1)\,\alpha'(1) = \int_0^1\int_0^Y \frac{W^2(Y)}{W^2(Z)}\,dY\,dZ. \tag{27}$$

Secondly, to make (23) satisfy the kinematical boundary condition $y = 1+\eta$ for $s = 1$, we have

$$1+\eta = \int_0^1 \frac{W(s)\,ds}{[W^2(s)-2A]^{\frac{1}{2}}} - \alpha(1)\,A''. \tag{28}$$

When (26) is substituted for A in (28) and terms in η'' (i.e. the smallest needing to be retained) are collected together, the result can be arranged as

$$g\{I_2(1)\,\beta(1)-\alpha(1)\}\,\eta'' = 1+\eta-\int_0^1 \frac{W(s)\,ds}{[W^2(s)-2g\eta]^{\frac{1}{2}}}. \tag{29}$$

To obtain a more easily interpretable equation for η, (29) is multiplied by η' and integrated, the arbitrary constant being determined by the condition $\eta' = 0$ for $\eta = 0$. The order of the double integration now arising on the right-hand side can be reversed, and so we get directly

$$g\{I_2\beta(1)-\alpha(1)\}\,\tfrac{1}{2}\eta'^2 = \eta+\tfrac{1}{2}\eta^2+\frac{1}{g}\int_0^1 W(s)\,[\{W^2(s)-2g\eta\}^{\frac{1}{2}}-W(s)]\,ds. \tag{30}$$

When the steps leading to this equation are reviewed, it is seen that the approximations have been concerned only with terms in the derivatives of η, and so far all operations involving η alone have been carried through exactly. Therefore, as a means to finding the maximum elevation of the wave (i.e. $\eta = a$ for $\eta' = 0$), this equation might still be useful even if $W^2(s)$ falls, in some part of the range $0 \leqslant s \leqslant 1$, to small values not much larger than $2g\eta$. However, proceeding on the assumption that this is not the case, we may reduce the integral in (30) by taking the first four terms in the binomial expansion of $[1-\{2g\eta/W^2(s)\}]^{\frac{1}{2}}$. Thus we obtain

$$\{I_2(1)\,\beta(1)-\alpha(1)\}\,\eta'^2 = \eta^2[\{g^{-1}-I_2(1)\}-gI_4(1)\,\eta], \tag{31}$$

where I_4 is the integral like (21) with integrand W^{-4} instead of W^{-2}. The error implied in this result is known to be $O(\eta^4)$ or $O(\eta''^2)$.

Equation (31) has the same form as the equation which gives a first approximation to the profile of the irrotational solitary wave (cf. Lamb 1932, §252, equation (11)). To confirm that in all cases possible at present the wave is one of elevation only, as it is in the irrotational case, it is necessary to prove that the coefficient of η'^2 in (31) is always positive: the proof is a simple matter, but is conveniently deferred to Appendix I. In consequence of this coefficient being positive, the factor between the brackets [ ] in (31) must be positive, and it follows that η varies between zero and a maximum value a given by

$$a = \frac{g^{-1}-I_2(1)}{gI_4(1)}. \tag{32}$$

When the origin of x is taken beneath the wave crest, the appropriate solution of (31) is

$$\eta = a\,\mathrm{sech}^2\,(x/b), \tag{33}$$

with

$$b = 2\left[\frac{I_2(1)\,\beta(1)-\alpha(1)}{g^{-1}-I_2(1)}\right]^{\frac{1}{2}}. \tag{34}$$

The last three equations are the principal deductions of this investigation. Equation (33) gives the general wave-form, which is unchanged by the presence of vorticity in the stream (cf. Lamb 1932, §252, equation (12)). Equation (32) gives the wave amplitude in terms of the wave speed c and the primary velocity distribution $U(y)$. It is seen that the functional dependence of a on c generally is far more complicated than in the irrotational case (U = const.) and is incapable of explicit inversion; however, as the more significant interpretation from a practical point of view, the equation can be regarded as an implicit relation for c in terms of a. Note that, for given a, equation (32) specifies two values of c, one necessarily positive value relating to propagation in the direction of the stream and another value, which may be negative, relating to propagation against the stream. Equation (34) gives the length scale of the solitary wave in terms of c and $U(y)$.

5. Discussion

First we consider the extreme case of an infinitesimal wave. According to (32), the condition $a \to 0$ implies that $I_2(1) = g^{-1}$ or, when this is written in full,

$$\int_0^1 \frac{dy}{[U(y)-c]^2} = \frac{1}{g}. \tag{35}$$

Further, (34) shows the length b to become indefinitely large; and so it is no surprise that (35) reproduces the formula obtained by Burns (1953) for the velocity of infinitesimal long waves. When $U(y) = U$ (a constant), (35) gives the result obtained from the usual linearized theory for long waves: thus

$$c = U \pm C_0, \tag{36}$$

where $C_0^2 = g$ (i.e. $C_0^2 = g^*h$ in dimensional units). That is, the two possible velocities of the wave relative to the stream are simply $\pm C_0$, for propagation in the downstream and upstream directions respectively. For general $U(y)$, (35) cannot be solved to give c explicitly; but Burns has established two significant general conclusions regarding the magnitudes of the values of c which satisfy (35). The first conclusion applies under the following restrictions on $U(y)$, which do not exclude most cases with practical interest: $0 \leqslant U(0) < U(1)$; $U'(y) > 0$ for $0 \leqslant y < 1$; $U''(y) < 0$ for $0 \leqslant y \leqslant 1$. It is that two values of c exist, one of which is greater than $U(1)$ and the other of which is less than $U(0)$.† Independently of the above restrictions, the second conclusion is that for non-uniform distributions $U(y)$, the wave velocity relative to the *mean* flow is always greater in magnitude than C_0, the value according to simplified theories ignoring the dynamical effects of vorticity.

It appears also that the relative speed of a finite solitary wave on a non-uniform stream is in general greater than the value given by potential theory. As a simple check on this property, let us take the case where the variation of $U(y)$ is small compared with the velocity, say $C = c - \overline{U}$, of the wave relative to the mean flow. Thus we write $W(y) = -C + \overline{U}\delta(y)$, where $\delta(y)$ is the fractional variation of $U(y)$ from its mean value $\overline{U}$, and we have $(\overline{U}/C)\,\delta(y) \ll 1$. Hence we obtain approximately, on expanding the integrand binomially,

$$\begin{aligned} I_2(1) &= \int_0^1 \frac{dy}{[C - \overline{U}\delta(y)]^2} = \frac{1}{C^2}\left\{1 + 3\left(\frac{\overline{U}}{C}\right)^2 \int_0^1 \delta^2(y)\,dy\right\} \\ &= C^{-2}\{1 + 3(\overline{U}/C)^2\,\Delta\}, \quad \text{say.} \end{aligned} \tag{37}$$

Here Δ is the mean square of δ, and we have used the fact that the mean of δ is zero by definition. Similarly we get

$$I_4(1) = C^{-4}\{1 + 10(\overline{U}/C)^2\,\Delta\}. \tag{38}$$

† I suspect that this second theoretical result, $c < U(0)$, is spurious when the stream is a long way supercritical—i.e. when $\overline{U} \gg C_0$. It is true that when $\overline{U}^2 \gg g$ a solution of (35) can always be found such that $\iota = \{U(0) - c\}/\overline{U} > 0$: the integral in (35) is then almost entirely comprised near the lower limit, and the equation becomes in effect $\iota^{-2} \sim \overline{U}^2/g$, so that a sufficiently small ι can balance any value of $\overline{U}^2/g$ however large. But the physical implications of this seem altogether 'unnatural'. The streamlines are implied to follow the wave contour everywhere except very close to the bottom, where the velocity variation due to the wave is consequently very large, and in reality this sort of behaviour would place a severe restriction on the wave amplitude if separation were to be avoided. Burn's theory was formulated in such a way as to rule out the possibility of separation, and the present result seems to be a concomitant of this artificial restraint rather than of any essential physical factor. It appears much more likely that when $\overline{U} \gg C_0$ a wave propagating against the stream will cause separation or some comparable effect indicated by a singularity in the linearized theory, and in fact the wave will be convected downstream at an absolute velocity not much different from $\overline{U} - C_0$.

Substitution of (37) and (38) into (32) leads to

$$a\left\{1+10\left(\frac{\overline{U}}{C}\right)^2\Delta\right\} = \left(\frac{C}{C_0}\right)^2\left\{\left(\frac{C}{C_0}\right)^2 - 1 - 3\left(\frac{\overline{U}}{C}\right)^2\Delta\right\}; \tag{39}$$

and, solving this equation for C^2 to the first order in a, we obtain finally

$$C^2/C_0^2 = 1+3F^2\Delta + a(1+4F^2\Delta), \tag{40}$$

where $F = \overline{U}/C_0$ is the Froude number of the stream according to the usual definition.

For a uniform velocity distribution, (40) reduces to

$$C^2/C_0^2 = 1+a, \tag{41}$$

which is Rayleigh's formula for the wave velocity (cf. Lamb 1932, § 252, equation (9)). Hence we see from (40) that the effect of vorticity in the stream is to increase the speed of a solitary wave above the 'classical' value given by (41). This increase comprises both an addition to the component independent of amplitude, as has already been shown by Burns, and also an augmentation of the influence of finite amplitude.

It appears, however, that the effect of vorticity on the wave speed is quite small in most instances having any practical bearing. In the first place, (40) shows the effect to be slight when the Froude number is small, as is commonly the case in large channels and rivers. Thus, on a stream flowing at only a small fraction of its critical velocity C_0, a solitary wave propagates at a speed which is scarcely affected by the vorticity present, even if at some depths the vorticity $U'(y)$ has quite high values. This could be argued on physical grounds; but it is far from immediately evident analytically that the application of potential theory is well justified in this case.

In the second place, the factor Δ is rather small for velocity distributions typical of real open-channel flows. Consider, for example, the empirical power law

$$U = (1+k)\,\overline{U}y^k, \tag{42}$$

which, with k about $\frac{1}{7}$, approximates quite closely to the velocity distributions observed in turbulent flows along smooth channels. For the distribution (42) we find that $\Delta = k^2/(1+2k)$; hence $\Delta = 1/63$ when $k = \frac{1}{7}$. With the latter value of Δ, the 'vorticity corrections' in (40) would hardly have any practical significance unless the Froude number were fairly large.

[Note incidentally that although (41) exactly reproduces Rayleigh's formula for the velocity of the irrotational solitary wave, the step from (39) to (41) entails an approximation which does not arise in Rayleigh's analysis. Writing $(C/C_0)^2$ for short as f, we recall that his result is derived as precisely $f = 1+a$, without the need arising to neglect terms which are explicitly $O(a^2)$, even though the apparent error is $O(\epsilon^4)$ which is actually equivalent to $O(a^2)$ (cf. Lamb, § 252). On the other hand, (39) gives $a = f(f-1)$, which leads to $f = 1+a-a^2-O(a^3)$. Our omission of second- and higher-order terms in (41) is justified, of course, as being consistent with the over-all approximation—more precisely, because (32), and hence (39), was established by extracting the non-zero root from the cubic expression on the right-hand side of (31), which was derived only to $O(a^3)$. Thus, in so far as they are both, strictly speaking, first-order approximations in a, Rayleigh's result and the present one have the same analytical status (as have also the various other first approximations to the speed of the solitary wave which are to be found in the literature—e.g. the result $f = e^a$ derived by Stoker (1957, §10.9)). However, because of the feature of Rayleigh's analysis mentioned above, there is some justification for extending the application of his formula $f = 1+a$ to waves for which a^2 is not so small as to be negligible; and indeed it has been recognized since well before the turn of the century that this formula approximates very closely to the observed speeds of even fairly large solitary waves. The success of Rayleigh's formula was explained precisely in the work of Long (1956). He showed the correct second-order approximation to be $f = 1+a-\frac{1}{20}a^2$, and in fact he derived some further coefficients of this expansion in a, which successively diminish in magnitude. Therefore, owing just to the smallness of the exact coefficients of a^2 and higher powers, extrapolation of $f = 1+a$ to moderately high values of a still gives a good approximation, certainly better than the spurious second-order approximation obtained here from (39), or than Stoker's result mentioned just above. In Appendix II to this paper, Rayleigh's method of analysis is adapted to the case of a solitary wave on a stream with *constant* vorticity, and the results serve as a check on the general results derived in §4. Since like its prototype this alternative analysis leaves an error which is only $O(\epsilon^4)$, no approximation explicitly $O(a^2)$ being required, the results obtained are presumably somewhat more accurate than those in §4. Unfortunately, Rayleigh's method serves only for this one rather artificial example of flow with vorticity: it is altogether useless for the general vorticity distributions dealt with above.]

We next consider the results of §4 with regard to the length of the wave. From (32) and (34) it follows that

$$ab^2 = 4\left\{\frac{I_2(1)\,\beta(1)-\alpha(1)}{gI_4(1)}\right\}. \tag{43}$$

For a uniform velocity distribution, we obtain very easily $I_2(1) = C^{-2}$, $I_4(1) = C^{-4}$, $\beta(1) = \frac{1}{2}$ and $\alpha(1) = \frac{1}{6}C^{-2}$; then (43) together with (41) gives

$$ab^2 = \tfrac{4}{3}(C/C_0)^2 = \tfrac{4}{3}(1+a). \tag{44}$$

This result confirms the statement made several times earlier that $ab^2 = O(1)$ for the irrotational solitary wave. When the velocity distribution is not uniform, the correction to the result (44) is roughly of the same magnitude as the extra terms in (40) as compared with (41); but it is actually a rather complicated matter to obtain a general estimate of (43) to the same accuracy as (40). While avoiding these complications here, we can still assert, as a fact fairly evident from (43) (also Appendix I), that ab^2 remains $O(1)$ for a variable $U(y)$, at least when the mean-square variation Δ and the Froude number are not exceptionally large. This check on the order of magnitude of ab^2 justifies the approximations on which the analysis towards the end of §4 was based.

Finally, the relationships amongst wave amplitude, speed and length will be evaluated in the case of a stream with a constant vorticity G. As was mentioned two paragraphs above, this example is analysed by a different method in Appendix II, providing a check on the general results established in §4. For simplicity we take only the case $U(y) = Gy$; however, cases in which the primary velocity distribution has the form $U(y) = U(0) + Gy$ are effectively covered also, since clearly $c - U(0)$ will enter the results for these cases in precisely the same way as c for the present case.

When $W = Gy - c$ is substituted in the definitions of I_2, I_4, α and β, the various integrals are easily evaluated and it is found that

$$\left.\begin{aligned} I_2(1) = \frac{1}{c(c-G)}, \quad I_4(1) = \frac{c(c-G)+\frac{1}{3}G^2}{c^3(c-G)^3}, \\ I_2(1)\beta(1) - \alpha(1) = \tfrac{1}{3}c^{-2}. \end{aligned}\right\} \tag{45}$$

Putting the first two of these expressions into (32) and reducing the result by means of the approximation $1/I_2(1) = c(c-G) = g + O(a)$, we obtain

$$a = \frac{c(c-G)-g}{g+\frac{1}{3}G^2}. \tag{46}$$

The mean velocity of the primary flow is $\overline{U} = \frac{1}{2}G$, and as before we write $C = c - \overline{U}$ and $C_0^2 = g$. Then (46) gives, without further approximation,

$$\left.\begin{aligned} C^2 &= C_0^2 + \overline{U}^2 + (C_0^2 + \tfrac{4}{3}\overline{U}^2)\,a, \\ \text{or} \qquad C^2/C_0^2 &= 1 + F^2 + (1 + \tfrac{4}{3}F^2)\,a \end{aligned}\right\} \tag{47}$$

with $F = \overline{U}/C_0$. The part of (47) which is independent of a agrees with a result found by Burns (1953, equation (33)) on the basis of his linearized long-wave theory.

It is interesting to observe that in this example the exact formula (47) is the same as the approximate formula (40), which was derived on the assumption that the fractional deviation $\delta(y)$ of $W(y)$ from its mean value is small. We have now $\delta(y) = 2y - 1$, and hence $\Delta = \frac{1}{3}$. Equation (47) is reproduced when this value of Δ is substituted into (40).

We next use the results (45) to evaluate the right-hand side of equation (43) for ab^2. Since (43) is established only as a first-order approximation in a, it is consistent to use the zeroth-order result $c(c-G) = g$ (see equation (46)) to simplify the right-hand side. Thus it is found directly that

$$ab^2 = \frac{4(c-G)^2}{3g+G^2}. \tag{48}$$

Again using $c(c-G) = g$ and also writing $G = 2\overline{U}$, $G^2/g = 4F^2$, we obtain from (48)

$$\tfrac{3}{4}ab^2 = \frac{1-2\overline{U}/c}{1+\frac{4}{3}F^2}. \tag{49}$$

Now, if the velocity of the stream is negligibly small compared with the wave velocity, the right-hand side of (49) becomes unity; and again this is the result which is found by potential theory when the dynamical effects of vorticity are ignored. Therefore, to reveal the effect of vorticity on the length of the solitary wave, for any given amplitude, one has merely to compare the value of the expression (49) with unity.

To find the ratio $\overline{U}/c$ ($= \lambda$, say) appearing in (49), we observe that the equation $c(c-G) = g$ can be rearranged to give $\lambda^2 + 2F^2\lambda - F^2 = 0$. Hence we have $\lambda = \pm F\sqrt{(F^2+1)} - F^2$, where the plus sign refers to downstream propagation ($c > 0$) and the minus sign to upstream propagation ($c < 0$). Thus (49) becomes

$$\tfrac{3}{4}ab^2 = \frac{1+2F^2 \mp 2F\sqrt{(F^2+1)}}{1+\frac{4}{3}F^2}, \tag{50}$$

where the alternative sign is minus for downstream and plus for upstream propagation. This result shows that, in the downstream case, $\frac{3}{4}ab^2 < 1$ for all $F > 0$, so that the effect of vorticity is to shorten the wave. In the upstream case, $\frac{3}{4}ab^2 > 1$ for all $F > 0$, so that vorticity lengthens the wave. For example, take $F = 0{\cdot}5$. Then it is found from (50) that $\frac{3}{4}ab^2 = 0{\cdot}29$ and $1{\cdot}96$, respectively, for downstream and upstream propagation. In the first case a solitary wave of given amplitude is shorter by a factor $\sqrt{(0{\cdot}29)} = 0{\cdot}54$ than it would be on a uniform stream with the velocity $\overline{U}$; and in the second case the wave is longer by a factor $1{\cdot}40$.

6. Conclusion

The foregoing analysis provides a general means for calculating the properties of a solitary wave on running water when the flow velocity has any given vertical distribution, provided only that the limitations due to stagnation effects as explained in § 2 do not arise. These limitations are avoided entirely whenever the stream is more than marginally subcritical, and otherwise relate only to the case of propagation against the stream (i.e. as in figure 1 (a) when the upstream propagation velocity $-c$ is so far reduced, by convection of the wave downstream, as to make $W(y) = U(y) - c$ vanish for some y in (0, 1)); moreover, the question of their arising at or even considerably above the critical condition is a somewhat arbitrary one, depending on how much velocity of slip at the channel bottom is allowed as a feature of the theoretical model for the primary flow. As was suggested earlier, however, the possibility of finding solutions representing flows which separate from the channel bottom under the wave is interesting in view of the likelihood that such flows do actually occur sometimes; and this aspect of the problem might well be worth pursuing, though methods rather different from the present would be necessary.

On the theoretical side, perhaps the main contribution of this paper is merely to have established that solitary waves can occur upon rotational flows under gravity as well as in the more idealized circumstances assumed by previous theories. This general issue has not been faced up till now, presumably owing to the lack of a tractable perturbation method which can account for waves of finite amplitude. The present method is believed to be novel, and it may be of interest as regards other possible applications.

On the practical side, it is not seriously proposed that the general formulae presented here might replace those familiar ones derived from potential theory which are used by hydraulic engineers to estimate solitary-wave properties.

Indeed, the main service of this analysis is to have shown precisely that the presence of vorticity in the stream has little effect on the wave in many circumstances typical of real open-channel flows, so that the results according to potential theory, which represents the stream as having a uniform velocity equal to the mean of the actual distribution, will often apply with good accuracy. It has occasionally been attempted (e.g. by Boussinesq 1877) to justify this application of potential theory by careful physical reasoning, but the matter is generally taken for granted rather carelessly in text-books on hydraulics. Nevertheless, there may well be cases where the effects of vorticity would be significant in relation to a practical estimate of the wave properties; and it is worth emphasizing that these effects become enlarged when a stream is near the critical condition. In fact, one important practical application of solitary-wave theory is to the large *standing* waves that often arise in just these circumstances. If the reach of the channel over which the nearly critical condition holds is fairly long, these waves are likely to take the form of a periodic train, but each wave—particularly the leading one—may approximate closely to a solitary wave (see Benjamin & Lighthill 1954).

We note finally that the present analysis would, with slight modification, account for a class of periodic waves of finite amplitude corresponding to the 'cnoidal' waves which were discovered by Korteweg & de Vries (1895) upon generalizing Rayleigh's analysis of the irrotational solitary wave (see Lamb 1932, §253; also Benjamin & Lighthill 1954). To develop a theory of these waves, the essential point of departure is the step from (29) to (30). At present the condition $\eta' = 0$ for $\eta = 0$ is imposed. If this condition were relaxed and the constant were left undetermined, the outcome would be an equation like (31) except that the cubic in η on the right-hand side would in general have three distinct roots: one particular value of the arbitrary constant would, of course, recover the present case characterized by the double root $\eta = 0$. The modified form of equation (31) has a solution $\eta = B+(A-B)\,\mathrm{cn}^2(x/L;k)$, where the modulus k of the elliptic function is proportional to the square root of the wave amplitude $A-B$, and where the constant L equals the wavelength divided by the period $2K(k)$ of the elliptic function (cf. Lamb 1932, §253; Benjamin & Lighthill 1954, §3). It appears a considerable task, however, to sort out the details of this result, such as to evaluate the constants A, B and L in terms of specific physical quantities, and the matter may suitably be left for subsequent study.

Appendix I

The object here is to show in general that

$$I_2(1)\,\beta(1) - \alpha(1) > 0,$$

when $I_2(1)$, $\alpha(1)$ and $\beta(1)$ are defined by (21), (24) and (27) respectively.

Since by definition $W(Z)$ is a real monotonic function of Z in the interval (0, 1), therefore

$$I_2(Y) = \int_0^Y W^{-2}(Z)\,dZ$$

increases monotonically with Y in (0, 1), and *a fortiori*

$$\beta(X) = \int_0^X I_2(Y)\, W^2(Y)\, dY$$

increases monotonically with X in (0, 1). Thus, writing $\gamma(X) = \beta(1) - \beta(X)$, we have that

$$\gamma(X) > 0$$

if $0 \leqslant X < 1$. Hence it is seen that the quantity

$$I_2(1)\,\beta(1) - \alpha(1) = \beta(1) \int_0^1 W^{-2}(X)\, dX\, dX - \int_0^1 \beta(X)\, W^{-2}(X)\, dX = \int_0^1 \gamma(X)\, W^{-2}(X)\, dX$$

must be positive, since the integrand $\gamma(X)\, W^{-2}(X)$ is positive over the range of integration.

Appendix II

Here we return to the case of a stream with constant vorticity, considered at the end of §5, and we outline an alternative analysis adapting the method used by Rayleigh (1876) and Lamb (1932, §252) for the irrotational solitary wave. It is assumed that $W = Gy - c$, where G is constant. Then, according to (5), $\zeta = G$ is the same everywhere in the flow, and we have

$$\psi = \tfrac{1}{2}Gy^2 + \psi_1, \tag{A1}$$

where ψ_1 is a harmonic function which vanishes on the bottom $y = 0$ and which, in view of (4), has the property $\psi_1 \to -cy$ for $x \to \pm\infty$. Hence, introducing the expression for a harmonic function used by Rayleigh, we may write

$$\psi = \tfrac{1}{2}Gy^2 + yf(x) - \frac{1}{3!}y^3 f''(x) + \ldots, \tag{A2}$$

where $f(x)$ is as yet arbitrary, to be determined by the boundary conditions at the free surface $y = 1 + \eta = \xi$, say. The analysis proceeds approximately by neglecting quantities which are $O(\epsilon)^4$; thus only the first two terms in this expansion of ψ_1 are retained.

In the present case the kinematical condition (2) takes the form

$$\psi(x, \xi) = \Psi(1) = \tfrac{1}{2}G - c. \tag{A3}$$

When (A2) is substituted into (A3) and the equation is solved for $f(x)$ to the order of approximation specified above, the result is

$$\begin{aligned} f(x) &= -\frac{c}{\xi} + \tfrac{1}{2}G\left(\frac{1}{\xi} - \xi\right) + \tfrac{1}{6}\xi^2 f'' \\ &= -(c - \tfrac{1}{2}G)\left\{\frac{1}{\xi} + \tfrac{1}{6}\xi^2\left(\frac{1}{\xi}\right)''\right\} - \tfrac{1}{2}G(\xi + \tfrac{1}{6}\xi^2\xi''). \end{aligned} \tag{A4}$$

Next, the dynamical boundary condition (7) gives

$$(c - G)^2 - 2g(\xi - 1) = (\psi_x^2 + \psi_y^2)_{y=\xi} = (f + G\xi)^2 - (f + G\xi)\,\xi^2 f'' + \xi^2 f'^2. \tag{A5}$$

Eliminating $f(x)$ from (A 5) by means of (A 4), we obtain

$$(c-G)^2-2g(\xi-1) = [c-\tfrac{1}{2}G(1+\xi^2)]^2\left(\frac{1}{\xi^2}+\frac{2}{3}\frac{\xi''}{\xi}\right)$$
$$-\frac{\xi'^2}{\xi}[\tfrac{1}{3}(c-\tfrac{1}{2}G)^2+\tfrac{1}{3}(c-\tfrac{1}{2}G)\,G\xi^2-\tfrac{1}{4}G^2\xi^2]. \qquad \text{(A 6)}$$

If this equation is multiplied by ξ' and integrated, the arbitrary constant being determined so as to make $\xi' = 0$ for $\xi = 1$, the result is

$$(c-G)^2(\xi-1)-g(\xi-1)^2 = c^2\left(1-\frac{1}{\xi}\right)-cG\left(\xi-\frac{1}{\xi}\right)$$
$$+\tfrac{1}{4}G^2\left(\tfrac{1}{3}\xi^3+2\xi-\frac{1}{\xi}-\frac{4}{3}\right)+\frac{1}{3}\frac{\xi'^2}{\xi}[c-\tfrac{1}{2}G(1+\xi^2)]^2,$$

or, on rearrangement,

$$[c-\tfrac{1}{2}G(1+\xi^2)]^2\xi'^2 = 3(\xi-1)^2\{c^2-cG-\tfrac{1}{12}G^2(\xi^2+2\xi-3)-g\xi\}. \qquad \text{(A 7)}$$

Since the approximations leading to (A 7) consist only in neglecting $O(\epsilon^4)$, no power of $\eta = \xi-1$ being neglected, this equation is presumably rather more accurate than the corresponding equation (31) for the solitary wave on a stream with general velocity distribution. As has been seen earlier, however, (A 7) again indicates that $O(\epsilon^2)$ is equivalent to $O(\eta)$; and accordingly it is consistent to approximate further the left-hand and right-hand sides of the equation, to $O(\eta^2\epsilon^2)$ and $O(\eta^3)$ respectively. Thus one obtains

$$(c-G)^2\eta'^2 = 3\eta^2\{c(c-G)-g-(g+\tfrac{1}{3}G^2)\,\eta\}. \qquad \text{(A 8)}$$

Comparing this with (31), we have that the wave amplitude is

$$a = \frac{c(c-G)-g}{g+\frac{1}{3}G^2}, \qquad \text{(A 9)}$$

in agreement with (46); and we find that

$$ab^2 = \frac{4(c-G)^2}{3g+G^2}, \qquad \text{(A 10)}$$

in agreement with (48).

REFERENCES

BENJAMIN, T. B. & LIGHTHILL, M. J. 1954 On cnoidal waves and bores. *Proc. Roy. Soc.* A, **224**, 448.

BIÉSEL, F. 1950 Etude théorique de la houle en eau courante. *Houille blanche*, **5**, 279.

BOUSSINESQ, J. 1871 Théorie de l'intumescence liquide appeleé onde solitaire ou de translation se propageant dans un canal rectangulaire. *Comptes rendus* **73**, 755.

BOUSSINESQ, J. 1877 Essai sur la théorie des eaux courantes. *Mem. pres. Acad. Sci., Paris*, Vol. 23.

BURNS, J. C. 1953 Long waves in running water. *Proc. Camb. Phil. Soc.* **48**, 695.

FRIEDRICHS, K. O. & HYERS, D. H. 1954 The existence of solitary waves. *Comm. Pure Appl. Math.* **7**, 517.

KEULEGAN, G. H. & PATTERSON, G. W. 1940 Mathematical theory of irrotational translation waves. *J. Res. nat. Bur. Stand.* **24**, 47.

KELLER, J. B. 1948 The solitary wave and periodic waves in shallow water. *Comm. Pure Appl. Math.* **1**, 323.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. *Phil. Mag.* (5), **39**, 422.

LAMB, H. 1932 *Hydrodynamics*, 6th ed. Cambridge University Press.

LONG, R. R. 1953 Some aspects of the flow of stratified fluids. *Tellus* **3**, 42.

LONG, R. R. 1956 Solitary waves in one- and two-fluid systems. *Tellus* **8**, 460.

McCOWAN, J. 1891 On the solitary wave. *Phil. Mag.* (5), **32**, 45.

PACKHAM, B. A. 1952. The theory of symmetrical gravity waves of finite amplitude. II. The solitary wave. *Proc. Roy. Soc.* A, **213**, 238.

PETERS, A. S. & STOKER, J. J. 1960 Solitary waves in liquids having non-constant density. *Comm. Pure Appl. Math.* **8**, 115.

RAYLEIGH, LORD 1876 On waves. *Phil. Mag.* (5), **1**, 257. (*Collected Scientific Papers*, Vol. 1, p. 251.)

STOKER, J. J. 1957 *Water Waves*. New York: Interscience.

THOMPSON, P. D. 1949 The propagation of small surface disturbances through rotational flow. *Ann. N.Y. Acad. Sci.* **51**, 463.

URSELL, F. 1953 The long-wave paradox in the theory of gravity waves. *Proc. Camb. Phil. Soc.* **49**, 685.

WEINSTEIN, A. 1926 Sur la vitesse de propagation de l'onde solitaire. *Rend. Accad. Lincei* (6), **3**, 463.